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Module

The seminar is part of the Module OBM ("Outils de Base Mathématiques", that is, "Basic Mathematical Tools"). In addition, the module will include a personal work about complex numbers, to be done later in the semester.

The prerequisites are:

- Very few things for the parts I and II. Only the basic functions and their properties.
- For part III, about integrals, you should be able to:
  - know by hearts the basic derivative and primitive functions
  - handle the composition of functions
  - find a primitive of a function of the form $f'(u) \times u'$
  - do an integration by parts

Note that this module is a prerequisite for all others that you will have in the year.

Motivation

Some of you may have considered mathematics as a set of strange formulas that you must learn or hide in your pocket calculator, then recite during an exam in order to deduce new strange formulas, which may gain you a few points on your final mark. The most cynical people may even think that the main purpose of studying mathematics is to become a math teacher and hence to keep alive a tradition of sadism.

These points of view are not completely true.

Mathematics is above all ***the art of doing conjectures, of trying to prove or contradict them***, for the purpose of ***understanding better*** the phenomena we work with and of ***forecasting*** their behaviors. From this point of view, all of us, like Mr. Jourdain, speak mathematics[1] all our life without knowing it! Mathematics seems to be a science which relies more on reasoning than on experimentation. This aspect is due to the way it is formalized: we start with properties that we admit as true, then we transform and combine these properties gradually, remaining in the "true" area, until we reach a final property which is still true and solves the initial question. This is the purely technical point of view. But the latter should not mask the experimental point of view: ***it is important to test ideas***, to keep going on if the first reasoning does not work (figure 1) or if it does not immediately lead to the final result. It is important to focus on particular cases before solving the most general case. Sometimes, this will highlight defective reasoning or a counterexample. And sometimes, this will help to understand the underlying architecture of the problem.

To go through all these steps of a mathematical thinking, it is important to have a good grasp of the math language and all its possibilities[2]: this will be the purpose of the first section, about logic.

Another aspect, which is often neglected, is the ability to represent problems and objects: graphically, geometrically, or with a short and clear statement. It is necessary to switch easily from one representation to another. One never knows which of them will stimulate the imagination for a more complex structure, for a solution or for a counterexample highlighting defective reasoning. This way of thinking is also necessary for communication: an engineer is often in a situation to present his ideas to people who do not know anything about the topic he's working on. Then a clear and short graphic explanation is much more effective (figure 2). We will work with these graphic representations and imagine several ways to draw some typical problems. We will discuss their advantages and their draw-backs.

[1] as well as English or French, you can ask your English or TE teacher.

[2] « What looks good is clearly stated
and the words to say it flow with ease. »
One more question for your English or T.E. teacher:-p

Figure 1: `The Shadoks, from Jacques Rouxel`



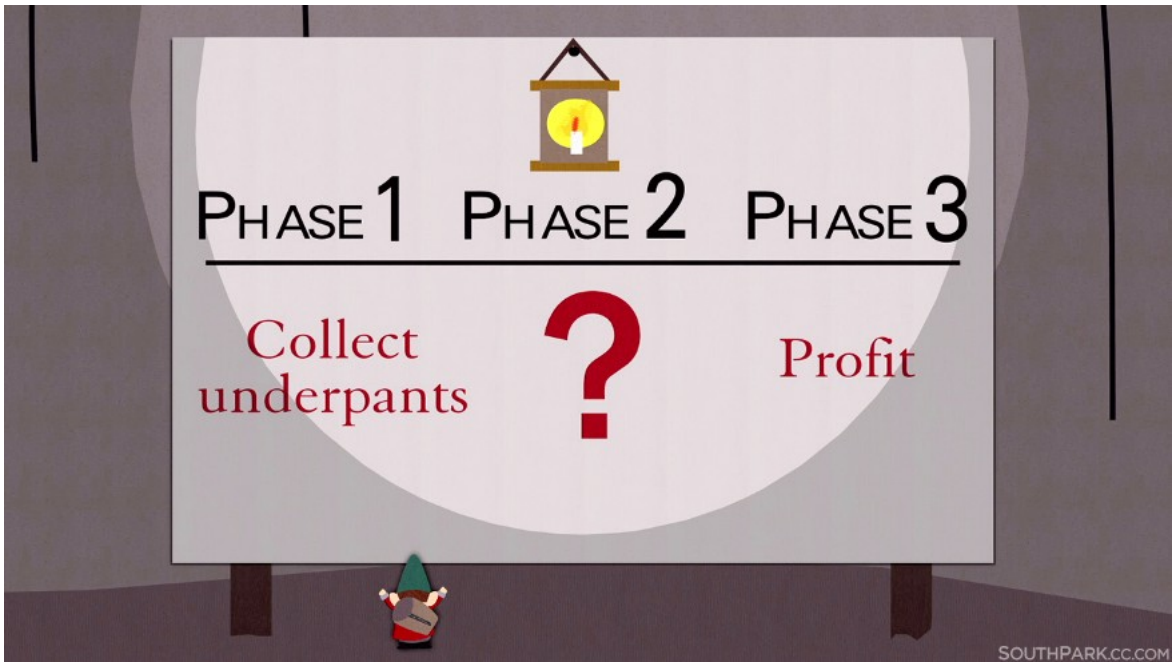


Figure 2: `A clear graphic can be much more effective to explain a reasoning... or to detect the mistakes in a plan which, at first, seemed to be perfect (SOUTH PARK: Gnomes business plan).`

We will also start working with basic examples of mathematical constructions. We will briefly introduce the vocabulary of sets and functions, echoing the statically typed language `CamL` that you are currently studying in algorithms. This is just one out of the many connections that you will see between the maths and algo classes. It will be frequent that a given question will be addressed from these complementary points of view. Indeed, what kind of engineer would be a computer scientist with no solid mathematical background? (see the answer on figure 3)

Finally, at the very last part of the seminar, we will study a mathematical tool that you have seen before: the integrals. It will be a break in the study of logic, less abstract, more focused on computations. You will need this tool, as well in maths as in other topics. It is used every time an area has to be computed.

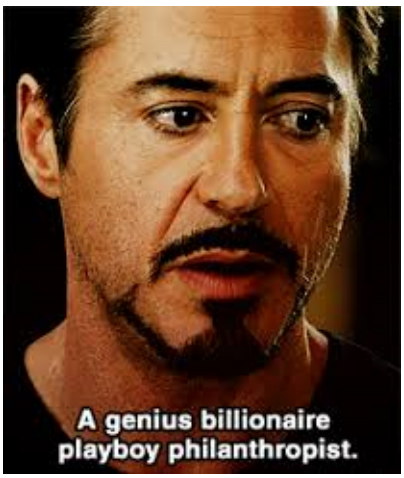


Figure 3: `Iron man's answer!`

Learning outcomes

The learning outcomes are the skills that we expect you to develop during the module, the skills that we will test at the exams. They are the following:

1. Logic
    - Recognize the logical connectors in a property
    - Use the "for all" and "there exists" quantifiers
    - Translate a property from the current language to the mathematical language and combine such properties to make a new one
    - Write the negation and the contrapositive of an implication, in spoken language as well as in mathematical language

2. Sets
    - Use the set vocabulary properly, especially the difference between the inclusion and membership symbols
    - Handle the operations involving sets and subsets: inclusion, intersection, union, complement, partition, cartesian product
    - For finite sets, compute the cardinal numbers of a set, a subset, a subsets union, a cartesian product and a power set

3. Reasoning
    - Prove a simple property by a direct proof, by contradiction or by induction
    - Prove a set inclusion or a set equality

4. Functions
    - Find the image or reverse image of a set by a function
    - Prove that a function (defined by a figure or a formula) is injective, surjective or bijective
    - Change the input or output set to make a function injective, surjective or bijective
    - Handle the arctan function as the reciprocal of the tan function. Draw its graph and find some particular values

- Test whether a relation is an equivalence or an order relation

5. Integrals
   - Compute an integral using a primitive function
   - Compute an integral using integration by parts, deciding by yourself what will be derivated and what will be integrated
   - Compute an integral using a substitution, the useful substitution being given

Some explanations

Teachers will say it many, many times: you will have to know your lecture by heart! Understanding and knowing your lecture has to be a priority. You will not succeed anymore the exercises, if you don't have a deep understanding of your lecture. This may be a change as compared to the previous years... the earlier you do it, the better.

Some exercises are labeled as "homework". They are additional exercises that you can do after the class, for training. Most of the times, they will not be corrected in the class.

The starred exercises ($\star$) are usually more difficult, more than what is required by the learning outcomes. They are bonus exercises: they are on the worksheet for those who want to go further. They will not be corrected in the class, but your teacher will be ready if you want to propose an idea for a solution.

Part I
Logic and sets

Logic, is the beginning of ***formalization***. This consists in translating properties, from intuitions or spoken language, into mathematical objects. Then these objects can be transformed and combined with some specified rules. To build this mathematical language, we need an alphabet (a set of characters), syntactic rules (different ways to combine the characters in order to express something) and some operations enabling one to alter the global meaning of the expressions.

Sets define the mathematical language's vocabulary. For example, defining the set $\mathbb{N}$ defines the "natural numbers". Sets are the most basic type of mathematical construction. In fact, most of the mathematical environments we will work with will be ***algebraic structures***, that is to say, sets enhanced by some available operations[3] between elements. A set, then, is a universe in which we can develop reasoning about the operations. But to start with, we will handle sets for themselves.

The syntax (logic) and the vocabulary (sets) that we study in this part results in the mathematical language. It will be used during the whole scholarship at EPITA, to write mathematical properties. During the seminar, you will train using it in the "Reasoning" section: you will have to prove in mathematical language that some properties are true or false.

1 Logic of propositions

1.1 Summary

The logic of propositions defines the syntactic rules of the mathematical language, that is, the different ways to combine basic propositions in order to produce a more complex one. This allows one to build very sophisticated sentences, using only basic elements and logical connectors. In a logical system accepting the law of excluded middle (we will remain in this framework), it is possible to get the Boolean value (true or false) of any complex proposition as a function of the Boolean values of its basic propositions.

Furthermore, you will see in the next paragraph, about sets, that all the logical connectors have a replication in terms of set operations.

1.2 Exercises

Exercise 1.1
Let x be a real number and consider the properties:

$$P : "x \leq 1", \quad Q : " -1 < x \leq 2" \quad \text{and} \quad R : "0 \leq x < 4"$$

1. If $x = -3$, which of the following properties are true: P, $\neg Q$, $\neg Q \vee R$, $\neg Q \wedge R$, $\neg P \wedge Q$, $\neg R \wedge P$?
2. **(Homework)** Same questions if $x = 3$.
3. Compare the domains of truth of $P \vee (Q \wedge R)$ and of $(P \vee Q) \wedge (P \vee R)$.
4. **(Homework)** Compare the domains of truth of $P \wedge (Q \vee R)$ and of $(P \wedge Q) \vee (P \wedge R)$.

[3] For example, the set $\mathbb{N}$ of the natural numbers may be enhanced by the operations of addition, multiplication, concatenation of the numbers written in the base 10 system, and so forth. A structure is defined by the set and by the available operations.

Exercise 1.2
The questions are mutually independent.

1. Write a necessary (maybe not sufficient) condition for the property "I got my baccalauréat diploma". Then find a sufficient (maybe not necessary) condition for the property "I got my baccalauréat diploma".

2. Let $x \in \mathbb{R}$. Replace each dotted line below with one of the symbols $\Longrightarrow$, $\Longleftarrow$ or $\Longleftrightarrow$:

   (a) $x^2 > 4$ ..... $x > 2$ **(b)** $x = \pi$ ..... $\sin(x) = 0$ **(c)** $\sqrt{x^2} = 1$ ..... $x = 1$

   (d) $\dfrac{x^2}{3} = 0$ ..... $x = 0$ **(e)** $e^{x+1} = 1$ ..... $x = -1$ **(f)** $x < 1$ ..... $|x| < 1$

3. Let $x \in \mathbb{R}$. Consider the property P : "$x^2 > 1 \Longrightarrow$ $(x < -1$ or $x > 1)$ ". Write the negation and the contrapositive of P.

Exercise 1.3
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

1. Translate the following properties into mathematical language, using quantifiers and symbolic notations:

   (a) f is bounded.
   (b) f never vanishes.
   (c) f is a constant function
   (d) f is periodic.
   (e) f is not the zero function.
   (f) f is positive on $\mathbb{R}^+$ (two translations: a first one with a quantifier, a second one with an implication).

2. Write the negation of all the properties above.

Exercise 1.4
For each of the properties below, say whether it is true or false. Then write its negation.

1. $\forall\, x \in \mathbb{R}, \quad \exists\, y \in \mathbb{R}, \quad x + y > 0$

2. $\forall x \in \mathbb{R}, \quad \forall\, y \in \mathbb{R}, \quad x + y > 0$

3. $\exists\, y \in \mathbb{R}, \quad \forall\, x \in \mathbb{R}, \quad x + y > 0$

4. $\exists\, x \in \mathbb{R}, \quad \exists\, y \in \mathbb{R}, \quad x + y > 0$

Exercise 1.5
Write the negation of the following sentences. Keep in mind that the word "some" means "at least one".

1. "This house is large"

2. "All truths are good to know"

3. "Some students sleep in maths class!"
4. "No criminal hypothesis can be discarded in this investigation"
5. "Some fishes are not red"
6. "If the weather is fine, then I go to the beach"
7. "You can go out if and only if you do your homework"
8. **(Homework)** "All the cats are grey"
9. **(Homework)** "Some students don't sleep in maths class"
10. **(Homework)** "If I go to bed late, then I sleep in maths class"
11. **(Homework)** "$\forall\, x \in \mathbb{R},\, \forall\, y \in \mathbb{R},\, x > y \implies x^2 > y^2$". Furthermore, say whether this property is true or false.
12. **(Homework)** "$\forall\, x \in \mathbb{R},\, \exists\, y \in \mathbb{R},\, y > x$". Furthermore, say whether this property is true or false.

2 Sets: definitions and operations

2.1 Summary

To start with, we review the different ways to define a set: extensional, intentional or parametric definitions. The latter consists in the formula defining the membership relation. The logical connectors, when applied to membership relations, can be extended to set operations: the logical negation results in the set complement, the logical disjunction in the set union, the logical conjunction in the set intersection. Thus, the computation rules about logical connectors and about set operations are the same.

Some more operations will be defined over sets. The "Cartesian product" of two sets results in a new set of a different type: it is a set of 2-tuples (ordered pairs).

The "power set" of a given set is the set of all its subsets. It is, therefore, a set of sets. We will use it when counting elements of subsets or in probability.

2.2 Exercises

Exercise 2.6

1. Compare the two following sets: $E = \{n \in \mathbb{N}, \exists\, k \in \mathbb{N}, n = 2k\}$ and $F = \{2p, p \in \mathbb{N}\}$.
2. Express the following sets in extension, in intention (using a membership function) and in parametric way (when it's possible):
   (a) Set of odd natural numbers, smaller than 20.
   (b) Set of natural numbers strictly between 2 and 30, and multiple of 4.
   (c) Set of real solutions of the equation $(x-1)(x^2-5x+6) = 0$.
   (d) Set of real numbers between -3π and 3π, whose sine is 0.
3. Express in spoken language the following sets. For each of them, give examples of some of its elements and some non-elements.

(a) $E = \{f : \mathbb{R} \longrightarrow \mathbb{R},\ \forall\, x \in \mathbb{R},\ f(x) \geq 0\}$.

(b) $F = \Big\{(u_n) \in \mathbb{R}^{\mathbb{N}}, \forall\, n \in \mathbb{N}, u_{n+1} - u_n \geq 0\Big\}$

(c) $G = \{x \in \mathbb{R},\ \exists\, y \in \mathbb{R}, x = y^2\}$

4. Let $A = [\![0, 10]\!]$. Replace each dotted line with one of the symbols $\in$, $\notin$, $\subset$ or $\not\subset$.

   a) -1 ... A, **b)** 2 ... A, **c)** $\emptyset$ ... A, **d)** $\{5\}$ ... A, **(e)** $\{3\}$ ... $\{1, 2, 3\}$, **(f)** $\{3\}$ ... $\mathcal{P}(A)$.

Exercise 2.7 ***(Homework)***

Consider the sets $A = \{x \in \mathbb{R}, -1 < x < 10\}$ and $B = \{n \in \mathbb{N},\ n + 1 \leq 8\}$.

1. Find a set S such that both A and B are subsets of S.
2. For each of the following properties, fill the dotted line with an expression such that the property is true (you are not allowed to fill the dotted line with $\emptyset$).

   Example : for $\cdots\cdots \in B$*, you can fill the dotted line with* 0 *because* $0 \in B$*.*

   (a) $\cdots\cdots\cdots \in A$ **(b)** $\cdots\cdots\cdots \notin A \cup B$ **(c)** $\cdots\cdots\cdots \subset A \cup B$ **(d)** $\cdots\cdots\cdots \in A \cap B$ **(e)** $\cdots\cdots\cdots \subset A \times B$.

3. What is the complement of B in $\mathbb{Z}$?

Exercise 2.8

Consider these three subsets of $\mathbb{R}$: $A = [0, 3]$, $B =]-1, 10]$ and $C = [2, 20]$.

Find $A \cap B$, $A \cup B$, $(A \cup C) \cap B$, $(A \cap B) \cup (C \cap B)$, $(A \cap C) \cup B$ and $\overline{A} \cup C$.

See the similarity with the logical connectors of section 1.

3 Finite sets

3.1 Summary

Finite sets are the easiest ones to study: it is possible to do an exhaustive enumeration (brute force enumeration) of all the possible situations one might face. Many algorithmic questions lead to counting the elements in a set. The difficult points are, first to determine the appropriate set, second, to count the elements. This will be the purpose of the next chapter, after this work session.

When we cannot get an exact count, it can be useful to find an order of magnitude (upper or lower bound) of the size of the set. For example, it may help to decide how many resources (in terms of processors or computation time) should be allocated to an algorithm's execution. This way of thinking is the starting point of many algorithmic discussions.

3.2 Exercises

Exercise 3.9

Consider the sets $E = \{1, 2, 3\}$ and $F = \{a, b\}$ (where a and b are the characters "a" and "b").

1. Express in extensions the sets $E \times F$ and $F \times E$. How many elements in these sets? How many elements in common?
2. Express in extensions the sets $\mathscr{P}(E)$ and $\mathscr{P}(F)$. For an arbitrary n-element set X ($n \in \mathbb{N}$), do you have an intuition about the number of elements in $\mathscr{P}(X)$?

Exercise 3.10

1. Let E and F be two finite disjoint sets. What is the value of $\text{Card}(E \cup F)$?

   Remark: when two sets E and F are disjoint, their union can be denoted: $E \sqcup F$.

2. Let E be a finite set. Remind what a "partition of E" is.

3. Let A and B be two finite non-disjoint sets.

   Represent graphically these two sets, and a partition of $A \cup B$.

   Using this partition, express the cardinal of $A \cup B$ using the cardinals of the partition, then the cardinal of A, B and $A \cap B$.

4. An EPITA class has 40 students. It has been observed in this class that:

   - Every student speaks at least one of the languages, English or Spanish.
   - 35 students speak English.
   - 20 students speak Spanish.

   How many students speak both English and Spanish?

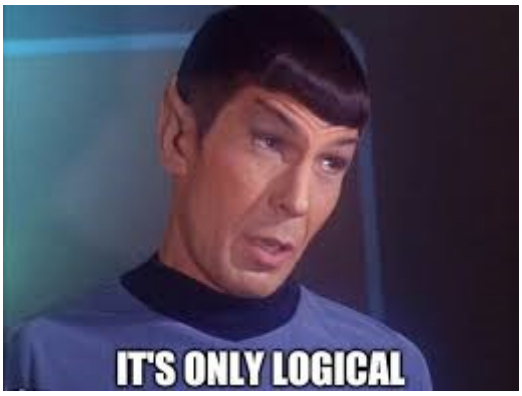
4 Reasoning

4.1 Summary

This section focuses on the different ways of reasoning which can be used to prove that a mathematical property is true. Most of the time, several ways may be considered to prove the property, so it is important to guess which one will be the easiest or the most convenient.

- The direct reasoning starts from the hypotheses and then uses a succession of implications to reach the desired property.
- The proof by contrapositive consists in proving the contrapositive of the desired property. Indeed, both properties are equivalent and, sometimes, the contrapositive is simpler to handle.
- The proof by contradiction consists in showing that, if the desired property was false, this would imply something that is obviously false. We can, therefore, deduce that the desired property is true.
- The proof by induction requires the proof of two sub-properties: first, the desired property is true in at least one particular case, and second, it propagates from this particular case to all other cases. We will mainly do proofs by induction on the set $\mathbb{N}$ of the natural numbers, but this reasoning can be extended to other sets. [4]

It is important to connect the latter reasoning with the recursive functions that you are studying in algorithmics. Proving a property by induction and writing a recursive function are very similar activities.



4.2 To write a nice proof redaction

Here are some redaction tips to prove properties.

Case you have to prove that: $\forall x \in E,\ P(x)$

The first line of the proof must be: Let $x \in E$ be given. Which can be understood in spoken language as: "let x be an unspecified element of E".

Then, for this $x \in E$, find a succession of implications which finally proves that $P(x)$ is true. If you get it, since x is an unspecified value in E, the property is true for all $x \in E$.

Case you have to prove that: $\forall x \in E, P(x) \Longrightarrow Q(x)$

The first line must be: Let $x \in E$ such that $P(x)$ is true.

[4] Of course, it is easy to extend the reasoning to $\mathbb{Z}$; you will see later some induction proofs on more complex sets, using different propagation processes.

Then find a succession of implications which finally proves that $Q(x)$ is true. Thus, if $P(x)$ is true, then $Q(x)$ is also true. That is, $P(x) \Longrightarrow Q(x)$.

Case you have to prove that: $\forall x \in E, P(x) \Longleftrightarrow Q(x)$

Most of the time, this is done in two steps. First, prove that $\forall x \in E, P(x) \Longrightarrow Q(x)$, then that $\forall x \in E, Q(x) \Longrightarrow P(x)$.

Case you have to prove a set inclusion: $A \subset B$

The sense of this is: $\forall\, x \in A, x \in B]$

The first line of the proof must be: Let $x \in A$ be given.

Then, for this $x \in A$, find a succession of implications which finally proves that $x \in B$. If you get it, you have proven that any unspecified element of A is in B, that is, all A-elements are in B.

Case you have to prove a set equality: $A = B$

Most of the time, this is done in two steps. First, prove that $A \subset B$, then that $B \subset A$.

Last but not least: a proof requires a redaction!! Specify the logical links between successive lines. These links can be mathematical symbols, or simple words, like "then", "since","thus", "hence", "therefore", etc...

4.3 Exercises

Exercise 4.11

1. Show that $\forall\,(x, y) \in]0, 1[^2$, $0 < \dfrac{x+y}{1+xy} < 1$.

   Hint: find the sign of $(1-x)(1-y)$, then expand this expression.

2. Show that: $\forall\, n \in \mathbb{N}$, (6 divides n) $\Longrightarrow$ (3 divides n). What do you think about the reciprocal?

3. Show that: $\forall\,(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 0 \iff x = y = 0$.

4. Let A be the set of the real constant functions and B the set of the real even functions. Show that $A \subset B$. What about the property $A = B$?

5. Let $A = \{(x, y) \in \mathbb{R}^2,\ 2x + y = 4\}$ and $B = \{(t-1, 6-2t),\ t \in \mathbb{R}\}$. Show that $A = B$.

Exercise 4.12

Let $n \in \mathbb{N}$. Using a proof by contrapositive, show that if n^2 is even, then n is even.

Exercise 4.13

Show by contradiction that $\dfrac{1}{3}$ is not a decimal number.

★ **Exercice 4.14**

Using the property from exercise 4-12, show that $\sqrt{2}$ is not a rational number.
Hint: you can use the fact that any positive rational number can be expressed as a ratio $\dfrac{p}{q}$, where p and q are positive natural numbers, and where these numbers are coprime (that is to say, they have no co-divisor other than ± 1).

Exercise 4.15

1. Prove by induction that: $\forall\, n \in [\![6, +\infty[\![,\ 2^n \geq 6n + 7$

2. Prove by induction that: $\forall\, n \in \mathbb{N}^*,\ \sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$

3. Consider the function $f : x \mapsto xe^x$ and accept without proof that it is infinitely differentiable on $\mathbb{R}$. Let $f^{(n)}$ $(n \in \mathbb{N})$ denote its n-th derivative. For example, $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)} = f'''$, etc.

   Prove that $\forall\, n \in \mathbb{N}$ and $\forall\, x \in \mathbb{R}$, $f^{(n)}(x) = (x + n)e^x$.

Exercise 4.16 ***(Homework)***

1. Show that: $\forall (x, y) \in \mathbb{Q}^2 \quad [x < y \Longrightarrow \exists\ q \in \mathbb{Q},\ x < q < y]$

2. Consider a real function f defined on $\mathbb{R}$ such that: $\forall\, x \in [0, 2],\ f(x) \in [1, 2]$.

   Let (u_n) be the sequence defined by: $\begin{cases} u_0 = 0.5 \\ \forall n \in \mathbb{N},\ u_{n+1} = f(u_n) \end{cases}$.

   Show by induction that: $\forall n \in \mathbb{N},\ u_n \in [0, 2]$

3. Let a be a positive real number. Prove by induction that: $\forall n \in \mathbb{N}^*, a^n + na^{n-1} \leq (a + 1)^n$.

4. Prove by induction that: $\forall\, n \in \mathbb{N}^*,\ \dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \ldots + \dfrac{1}{n \times (n+1)} = 1 - \dfrac{1}{n+1}$

Part II
Functions

The rigorous definition of a function from an input set E to an output set F is a subset of $E \times F$ fulfilling some specific properties. Yet, it is more intuitive to see functions as processes which transform elements from E into elements from F, or as a mapping between elements from both sets. You have started, in `CamL`, to program objects called "functions": they take an argument (an element from the input set) and return a result (an element from the output set).

Relations between elements in a set E (for example, the "less than or equal" relations between real numbers, the "inclusion" relation between sets) are Boolean functions defined on $E \times E$. In the "Relations" section, we will focus on two types of relations: order relations and equivalence relations. Both enable one to define a graph structure on the set E.

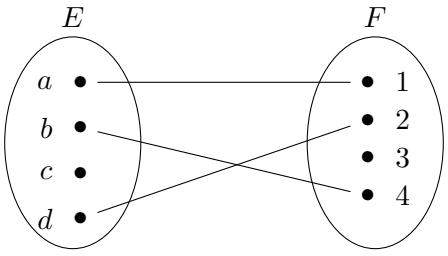
5 Image and reverse image

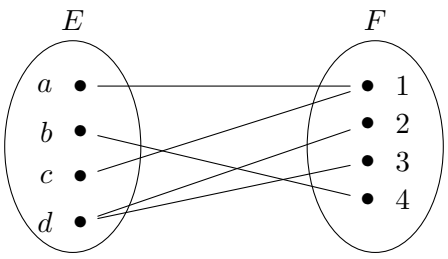
5.1 Summary

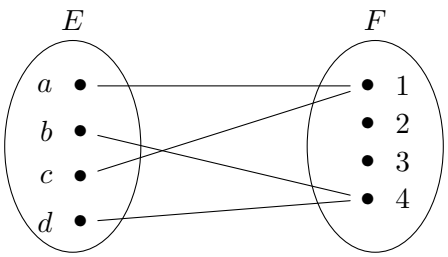
The notions of image and reverse image, which you have applied before to elements, can be extended to sets. Take care with the confusing notations: when A is a subset of the input set, $f(A)$ does not denote "the function f applied to the subset A", but "the set of all the values that f can take when applied to elements of A". Similarly, $f^{-1}(B)$ is defined for any subset B of the output set, even when the function f^{-1} does not exist. For example, the notations f and f^{-1}, when applied to sets, cannot be composed as they can be when applied to elements.

5.2 Exercises

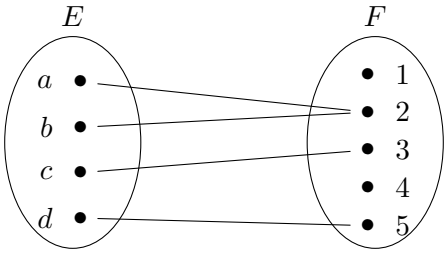
Exercise 5.17
For each of the figures below, say whether it represents a function from E to F.







Exercise 5.18
Let $f : \{a, b, c, d\} \longrightarrow \{1, 2, 3, 4, 5\}$ be the function defined by the following figure:



Find: $f(\{a, b, c\})$, $f(\{a, b, c, d\})$, $f^{-1}(\{2, 3\})$ and $f^{-1}(\{1, 5\})$.

Exercise 5.19
For each of the following properties, define a function (using a figure) from $[\![1, 4]\!]$ to $[\![1, 5]\!]$ such that the property is fulfilled.

1. $f([\![1, 4]\!]) = \{1, 4\}$
2. $f^{-1}(\{3, 4\}) = \{1, 2\}$
3. $f(\{1, 2, 3\}) = \{3\}$ et $f^{-1}(\{3, 5\}) = \{1, 2, 3\}$
4. $f^{-1}(\{2\}) = \{2, 3\}$ et $f(\{1, 2\}) = \{1, 2\}$

Exercise 5.20
Consider in $\mathbb{R}^{\mathbb{R}}$ the two functions $f : x \longmapsto \mathrm{e}^x$ and $g : x \longmapsto x^2$ and, on $[-\pi, \pi]$, the function $h : x \longmapsto \sin(x)$.

1. Draw the graphs of these functions.
2. Find the sets:
   - $f(\mathbb{R})$ and $f^{-1}(\{0\})$;
   - $g([-1, 4])$ and $g^{-1}([-1, 4])$;
   - $h([-\pi, \pi]), h([0, \pi/2]), h^{-1}([0, 1]), h^{-1}([1, 2])$ and $h^{-1}([3, 4])$.

Exercise 5.21 ***(Homework)***
Complete the following table.

N.B.: it is a good idea to draw, before answering, the functions' graphs on your draft!

f	$\exp : \mathbb{R} \longrightarrow \mathbb{R}$	$\sin : [-\pi, \pi] \longrightarrow \mathbb{R}$	$\ln :]0, +\infty[\longrightarrow \mathbb{R}$
$f(\{0\})$			
$f(]0, \pi])$			
$f^{-1}(\{0\})$			
$f^{-1}(\{1\})$			
$f^{-1}(]-\infty, 0])$			

6 Injective and surjective functions

6.1 Summary

Injectivity and surjectivity are two properties about functions, but which are very different. The fact that we study these properties together should not lead you to believe that one is the negation of the other.

A function is said to be injective if no pair of input values can lead to the same output value. Thus, from the value of $f(x)$ we can deduce the value of x and the function is, hence, reversible. Injectivity is closely related to the input set: any function can be transformed into an injective function by restricting the input set to remove the duplicates, that is, by keeping for each image only one reverse image.

A function is said to be surjective if any element of the output set is the image of an input element. Then the whole output set is the image of the input set, it has no unused value. For example, this is a good property for a hash table, the aim of which is to spread the input values into many output cells. Surjectivity is related to the output set.

A function which is both injective and surjective is bijective. Such a function implements an invertible transformation between the input and the output sets. We will see that the existence of such a function implies a strong relation between these sets.

6.2 Exercises

Exercise 6.22

Discuss the injectivity and surjectivity of the function $f : \left\{ \begin{array}{ccc} E & \longrightarrow & F \\ x & \longmapsto & x^2 \end{array} \right.$ in the following situations:

1. $E = \mathbb{R}$ and $F = \mathbb{R}$
2. $E = \mathbb{R}$ and $F = \mathbb{R}^+$
3. $E = \mathbb{R}^+$ and $F = \mathbb{R}$
4. $E = \mathbb{R}^+$ and $F = \mathbb{R}^+$

Exercise 6.23

1. Using a figure, define an injective function from $[\![1, 4]\!]$ to $[\![1, 5]\!]$. Can this function be surjective?
2. Using a figure, define a surjective function from $[\![1, 4]\!]$ to $[\![1, 3]\!]$. Can this function be injective?
3. Getting your intuitions from the previous questions, define a bijective function.

Exercise 6.24

1. Let f, g and h be the functions of $\mathbb{R}^{\mathbb{R}}$ defined for any $x \in \mathbb{R}$ by:

   $$f(x) = x^2 \quad , \quad g(x) = x + 1 \quad \text{and} \quad h(x) = \cos(x)$$

   Write explicit formulas for the functions $f \circ f, f \circ g, g \circ f$, $g \circ g$, $g \circ h$ and $h \circ g$.

2. In $\mathbb{R}^{\mathbb{R}}$, consider the functions $f : x \longmapsto 2x$, $g : x \longmapsto -2x$ and $h : x \longmapsto x^2$.
   Compare $h \circ f$ with $h \circ g$. What can you deduce from that?

3. Consider two functions $f : [0, 1] \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}^+$. Which composition(s) is(are) possible?

Exercise 6.25

Let E and F be two sets and f a function from E to F. Show that:
f bijective $\iff \forall\, y \in F, \exists !\, x \in E$ such that $y = f(x)$.

We accept without proof the following theorem.

Theorem: Let E and F be two sets and $f \in F^E$. Then,
f bijective $\iff \exists\, g : F \longrightarrow E$ such that $\forall\, x \in E$, $g \circ f(x) = x$ and $\forall\, y \in F$, $f \circ g(y) = y$.

If such a function g exists, then is is unique and called the "reciprocal function of f" (denoted by $g = f^{-1}$). Thus, for all bijective function f from E to F:

$$\left\{ \begin{array}{rcl} y & = & f(x) \\ x & \in & E \end{array} \right. \iff \left\{ \begin{array}{rcl} x & = & f^{-1}(y) \\ y & \in & F \end{array} \right.$$

Furthermore,

$$\forall\, x \in E,\ f^{-1} \circ f(x) = x \ \text{ and } \ \forall\, y \in F,\ f \circ f^{-1}(y) = y$$

The graphs of f and f^{-1} are symmetric regarding the line of equation $y = x$.

Application: the arctan **function. To remember by heart.**

1. Explain why the function tan is bijective from $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ to $\mathbb{R}$.
2. Consider its reverse function $\tan^{-1}$, which is denoted by arctan.
   (a) Draw the graph of this function.
   (b) Show that $\forall y \in \mathbb{R}$, $\arctan'(y) = \dfrac{1}{1+y^2}$.
   (c) Fill the following table:

α	0		$\frac{\pi}{3}$			$-\frac{\pi}{4}$	$\arctan(x)$
$\tan(\alpha)$		1		$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$		x

3. Find other pairs of bijective functions which are reciprocal of each others.

★ **Exercice 6.26**

Consider three sets E, F and G and two functions $f : E \longrightarrow F$ and $g : F \longrightarrow G$.

1. Show that if f and g are injective, then $g \circ f$ is injective.
2. Show that if f and g are surjective, then $g \circ f$ is surjective.

7 Relations

7.1 Summary

Relations between elements in a set E are Boolean functions defined on $E \times E$. They can be represented as oriented graphs. In this part, we will focus on two types of relations which fulfill some specific properties. Their handling is simplified consequently.

Equivalence relations split the set E into a partition of subsets, where the elements in each subset are mutually connected. The idea is to classify the elements in boxes, in such a way that all elements in a given box share a property defined by the relation. Such relations will be used in arithmetics (congruence) and in linear algebra (when defining matrix reductions).

Order relations enable one to compare the elements in a set. Some of them are «total order relations»: any pair of elements can be compared and a global ranking is possible. Other order relations are only «partial». Order relations are omnipresent, in mathematics as well as in algorithmics: comparison and sorting of elements, topology, flow graphs, optimization problems and so forth.

7.2 Exercises

Exercise 7.27

Let us define, on the set $[\![0, 7]\!]$, the relation[5] R by:

$$\forall (x, y) \in [\![0, 7]\!]^2,\ x\ R\ y \iff (x - y) \text{ is a multiple of } 3$$

1. Show that R is an equivalence relation.
2. Draw its graph. What do you notice?

Exercise 7.28

1. (a) Show that the $\leqslant$ relation is a total order relation over $[\![0, 5]\!]$.
   (b) Draw its graph and its Hasse diagram.
2. (a) Show that the $\subset$ relation is a partial order relation over $\mathscr{P}(\{1, 2, 3\})$.
   (b) Draw its graph and its Hasse diagram.

★ **Exercice 7.29**

Using the Latin alphabet $E = \{a, b, \ldots, z\}$, let us define the set E^* of all the words [6] writable with these letters, of any length.

For the following relations over E^*, say whether they are total order, partial order or equivalence relations (w and w' denote two arbitrary words):

1. $w\ R_1\ w'$ if the words w and w' have the same length;
2. $w\ R_2\ w'$ if w would be ranked before w' in a dictionary;
3. $w\ R_3\ w'$ if w is a sub-word of w', that is, if we can find two words u and v (maybe empty) such that w' is the concatenation $u \cdot w \cdot v$.

   For example: **logic** is a sub-word of *psycho***logic***altrauma*[7].

[5] This relation is the congruence modulo 3.

[6] That is, character strings.

[7] In English, we should write «psychological trauma», with a blank character. Note that the character strings that we use in the exercise do not need to be existing English words. They are just arbitrary sequences of letters.

Part III
Integration

8 Primitive functions, integration by parts, substitution

8.1 Summary

Many centuries ago, mathematicians have started computing areas of geometric figures, such as regular polygons. In that purpose, they used to split these figures in small triangles or rectangles. Then they addressed the question of finding the areas of less regular figures, delimited by the graph of some arbitrary functions. This was the beginning of the integration notion.

The integral of a continuous function f between two bounds a and b is the area delimited by the (Ox) axis, the graph of the function and the vertical lines of equations $x = a$ and $x = b$. You have seen before an approximation method using rectangles: the interval $[a, b]$ is split into small sub-intervals. Then the areas of rectangles just "below" and just "above" the graph of f in each sub-interval are computed. The smallest the sub-intervals, the more precise the approximation.

For an exact computation of the area, we can use primitive functions (antiderivatives). The integral of a continuous function f between two bounds a and b is the difference $F(b) - F(a)$, where F is a primitive function of f.

Sometimes, no direct computation can be done because no primitive function can be found. Then there are other methods: integration by parts (a formula resulting from the derivative of a product of functions) or integration by substitution.

Integrals are useful in many mathematical topics. In probability, they are used when handling continuous random variables, to compute an expectation, a variance or a cumulative distribution function. In signal analysis, they enable one to find the Fourier transform of a signal. Furthermore, integrals are used in many other scientific topics. In physics, they enable one to compute lengths, areas, volumes or flux. They also can be used in medical science, to study a population evolution or a medicine propagation in the blood for examples. There are many other topics.

8.2 Exercises

Exercise 8.30

Find the derivative of the functions f, g and h (no need to discuss the definition domain).

$$f(x) = \sin^{42}\Big(\ln\big(x\sin(x)\big)\Big) \qquad g(x) = \sqrt{\ln\Big(e^{x^2}+1\Big)} \qquad h(x) = \ln\Big(2^{\,x} - \sin\big(\sin(x)\big)\Big)$$

Exercise 8.31

Using primitive functions, compute the following integrals.

1. $I_1 = \displaystyle\int_0^2 3te^{-t^2}\,\mathrm{d}t.$
2. $I_2 = \displaystyle\int_0^1 4t(2t^2+1)^4\,\mathrm{d}t.$
3. $I_3 = \displaystyle\int_1^e \frac{(\ln(t))^2}{t}\,\mathrm{d}t.$
4. $I_4 = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan(x)\mathrm{d}x$
5. $I_5 = \displaystyle\int_0^1 \frac{t^2+1}{2\sqrt{t^3+3t}}\,\mathrm{d}t.$
6. $I_6 = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\sqrt{\cos(x)}}\,\mathrm{d}x.$
7. $I_7 = \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan(x)(1+\tan^2(x))\mathrm{d}x$
8. $I_8 = \displaystyle\int_0^1 \frac{2x^3+1}{(x^4+2x+1)^3}\mathrm{d}x$
9. $I_9 = \displaystyle\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x$

Exercise 8.32
Remind the integration by parts theorem and prove it.

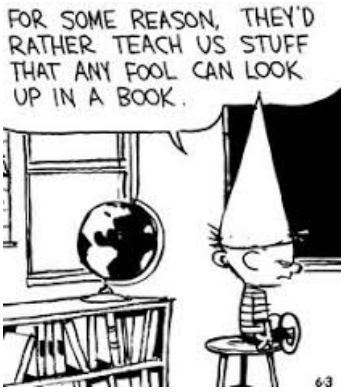
Exercise 8.33
Using integration by parts (IBP), compute the following integrals.

1. $I = \int_0^1 (2x+3)e^{3x}\,\mathrm{d}x$.
2. $J = \int_0^{\frac{\pi}{2}} x\cos\left(\frac{x}{2}\right)\,\mathrm{d}x$.
3. $K = \int_1^e \ln(x)\,\mathrm{d}x$.
4. $L = \int_0^{\pi} x^2\sin(x)\,\mathrm{d}x$.
5. $M = \int_0^{\pi} e^x\sin(x)\,\mathrm{d}x$.
6. ★ $N = \int_1^e \frac{\ln(x)}{x^2}\,\mathrm{d}x$.
7. ★ $O = \int_0^{\frac{\pi}{4}} \frac{x}{\cos^2(x)}\,\mathrm{d}x$.

Exercise 8.34
Using substitutions, compute the following integrals:

1. $I = \int_0^{\frac{1}{2}} \frac{4x^2}{(2x+1)^2}\,\mathrm{d}x$ using the substution $t = 2x+1$
2. $J = \int_0^{\ln(\sqrt{3})} \frac{\mathrm{d}t}{e^t + e^{-t}}$ using the substution $x = e^t$
3. $K = \int_1^e \frac{\ln(x)}{\sqrt{x}}\,\mathrm{d}x$ using the substution $t = \ln x$
4. $L = \int_{-1}^1 \frac{\mathrm{d}x}{(x+3)\sqrt{x+2}}$ using the substution $t = \sqrt{x+2}$
5. $M = \int_0^4 e^{\sqrt{x}}\,\mathrm{d}x$ using the substution $t = \sqrt{x}$
6. ★ $N = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin(t)\cos(t)}\mathrm{d}t$ using the substution $x = \tan(t)$

Bibliography



There are very few mathematical books with a point of view adapted to programmers. Fortunately, there are many more books written by computer scientists with a solid mathematical background. Here are some examples:

- *The Art of Computer Programming*, from Donald Knuth, is certainly the one containing the widest mathematical knowledge for a computer scientist; the author explains how mathematics can lead to efficient programming.
- *Introduction to Algorithms*, from Thomas E. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, is more algorithmics oriented. Still, it includes many discussions highlighting the interest of the mathematical notions.

Nevertheless, you should feel free to look for references by yourself and there are many. For most of your forthcoming projects, a first step will consist in building an adapted bibliography: it is much more effective to take time to understand the problem and the existing solutions, than to start programming immediately and then to realize that the idea was not the good one.